THE ORIGINS OF QUANTIZATION IN THE UNIVERSE

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Abstract: The Universe is quantized simply because its age is very long, so its cycle frequency is very small, but not zero! From this, the quantizations of all physical quantities can be derived.

Contents:

Chapter 1: Quantization and Indetermination from the Universe.

Par. 1.1: Introductory concepts.

If the world had ever existed, then what is happening now should have already happened.
A. SCHOPENHAUER.

If an event, after having had at its disposal an infinite time, hasn't happened yet, then it's because it can never happen.
In physics an infinite time is meaningless. The infinite is something you can just say and you can assign a symbol, but it can be neither imagined nor really handled.
In mathematics they talk about a tendency to infinite; just a tendency. The Universe cannot be born an infinite time ago; and so, what was before it? Well, we cannot say there isn't any answer, but rather we can say this question is wrong. Time was born together with the Universe and in the Universe, so the expression "before the Universe" is a contradiction. It exists since the moment when it started to exist and that's it. Or better, it exists and that's it. Rather, there is something more interesting: to understand how the Universe can "appear" without violating the conservation laws and laws of physics in general.
Well, we have to admit that if matter shows mutual attraction as gravitation, then we are in a harmonic and oscillating Universe in contraction towards a common point, that is the center of mass of all the Universe. As a matter of fact, the acceleration towards the center of mass of the Universe and the gravitational attractive properties are two faces of the same medal. Moreover, all the matter around us shows it want to collapse: if I have a pen in my hand and I leave it, it drops, so showing me it wants to collapse; then, the Moon wants to collapse into the Earth, the Earth wants to collapse into the Sun, the Sun into the centre of the Milky Way, the Milky Way into the centre of the cluster and so on; therefore, all the Universe is collapsing. Isn't it?
So why do we see far matter around us getting farther and not closer? Easy. If three parachutists jump in succession from a certain altitude, all of them are falling towards the center of the Earth, where they would ideally meet, but if parachutist n. 2, that is the middle one, looks ahead, he sees n. 1 getting farther, as he jumped earlier and so he has a higher speed, and if he looks back at n. 3, he still sees him getting farther as n. 2, who is making observations, jumped before n. 3 and so he has a higher speed. Therefore, although all the three are accelerating towards a common point, they see each other getting farther. Hubble was somehow like parachutist n. 2 who is making observations here.
At last, I remind you of the fact that recent measurements on far galaxies type Ia supernovae, used as standard candles, have shown an accelerating Universe; this fact is against the theory of our supposed current post Big Bang expansion, as, after that an explosion has ceased its effect, chips spread out in expansion, ok, but they must obviously do that without accelerating.
Sometimes, someone says that for two parachutists who are perfectly parallel each other, there wouldn't be any getting farther. Well, that's a limit situation in which the exception proves the rule. In the Hubble's Law for the expansion of the Universe, you cannot even number the exceptions (http://vixra.org/pdf/1206.0068v1.pdf).
Anyway, as the world wasn't born an infinite time ago, collapsing matter cannot come from an infinite distance; therefore, hundreds of billions years ago there was an expansion (post Big Bang), in the opposite direction with respect to the collapse we have now, and so all that with a repulsive gravity. On the basis of all that, the Universe is cyclic and so it

has a cyclic frequency **and this is the right key to understand why it is quantized!** All the frequencies which are in the Universe must so be, directly or indirectly, a multiple of the Universe one and this one is the smallest existing frequency.

Par. 1.2: The Planck/Einstein Equation and the quantization.

Planck/Einstein Equation $E = h\nu$ (and $E_T = nh\nu$, in case of many photons) tells us the energy of a photon is ν (frequency) times the energy box h (in joule); it is held somehow as the father of quantum physics, of energy boxes etc.

Before, we got such a special constant h from visual reasonings about the Universe and particles, but in the last century it appeared through the Planck/Einstein Equation, mainly through two separate phenomena: one, the Photoelectronic Effect, was studied mainly by Einstein, while the other, that is the studies on the Black Body Radiation Spectrum, was mainly treated by Planck. On the opinion of who is writing, here, both Einstein and Planck didn't intuite in advance their equation and the quantization, but rather were forced by circumstances to such suppositions in order to just make the theoretical interpretation match the results from the experiments!

Moreover, as a quantum is not as small as zero, but it has its own size somehow, then, in the opinion of who is writing, here, in the evaluation of physical quantities, uncertainties cannot be zero (The Heisenberg Uncertainty Principle, Schrödinger's Equation etc). If you see a particle, in order to figure out its position, you must interfere with it somehow, although through the smallest quantum of energy, and so you "touch" it, so you move it, so you change what you are going to measure.

In thermodynamics, too, where quantum physics acts deeply, if, for instance, I try to make a liquid in a calorimeter reach the absolute zero, I'll put a thermometer inside and start cooling as well as I can, through a refrigerator, but whenever I decide to check the temperature reached, in order to see if the absolute zero has been reached, then, in the opinion of the writer, I have to see the thermometer, so I have to illuminate it, although through just the smallest quantum of luminous energy, and so I heat it and it transmits some heat to to the liquid and therefore I'll never get the absolute zero.

Now, let's analyse both the above mentioned phenomena: the Photoelectronic Effect and the Black Body Radiation Spectrum.

Chapter 2: The birth of Quantum Physics.

Par. 2.1: The Photoelectric Effect and the walk to quantization.

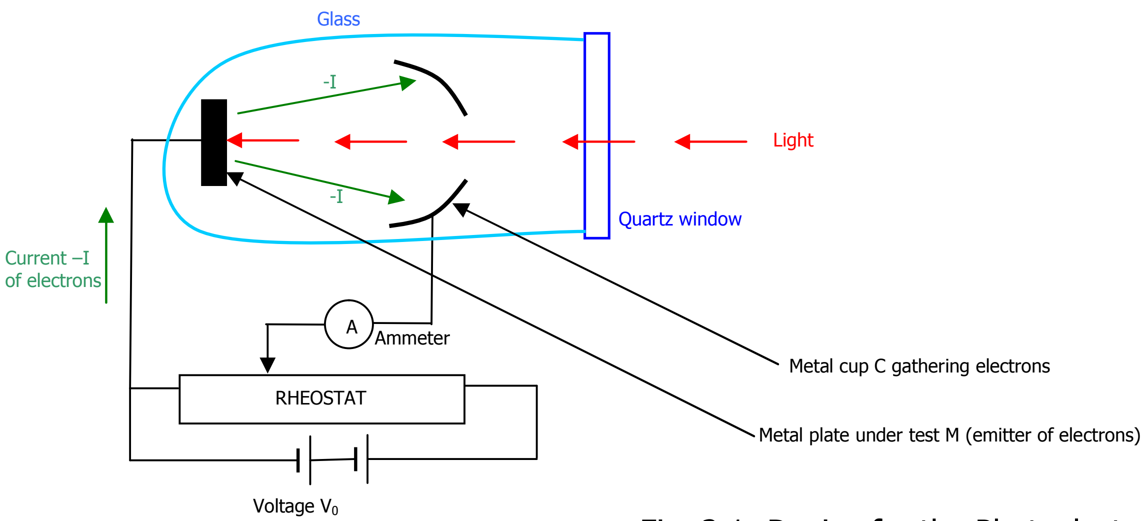

Fig. 2.1: Device for the Photoelectronic Effect.

Let the voltage between the cup C and plate M be: $\Delta V = V_C - V_M$ and let I be the current measured by the ammeter. Then, let I_∞ be the saturation current, that is the maximum current you can have with a certain light flux Φ.
From the experiments, we have:

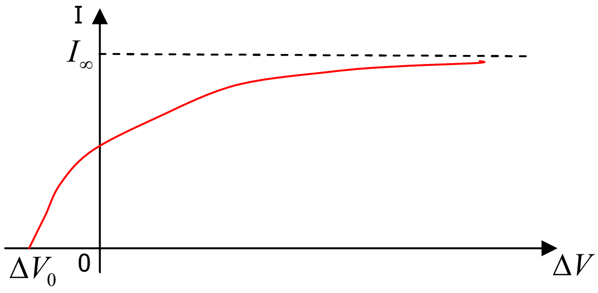

Fig. 2.2: Voltage-current graph.

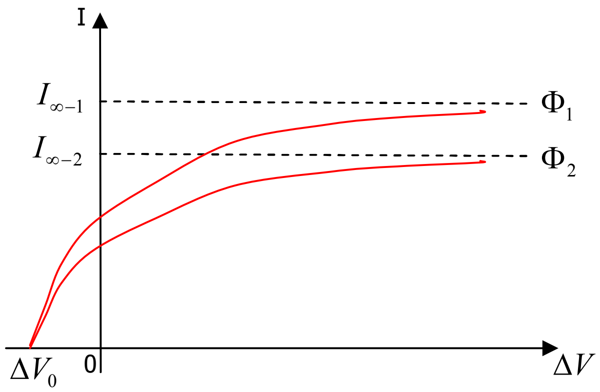

Fig. 2.3: Voltage-current graph for different light fluxes Φ.

Incident light makes electrons jump out of the plate M, and they are then gathered by the cup C, and accelerated, too, by a voltage.
We have that the electrons are emitted with a kinetic energy E_K which can be measured by supplying an inverse $\Delta V = \Delta V_0$ (stop voltage) so that the current of electrons emitted also with $\Delta V = 0$ is reduced to zero; when this happens, we have: $-e\Delta V_0 = E_K$.
From the experiments, we see that $\Delta V_0 \neq f(\Phi)$, that is: ΔV_0 does not depend on Φ, but, on the contrary, it depends on the frequency ν of the incident light.
All this is in a complete disagreement with classic physics.
The experiments show what is in Fig. 2.4:

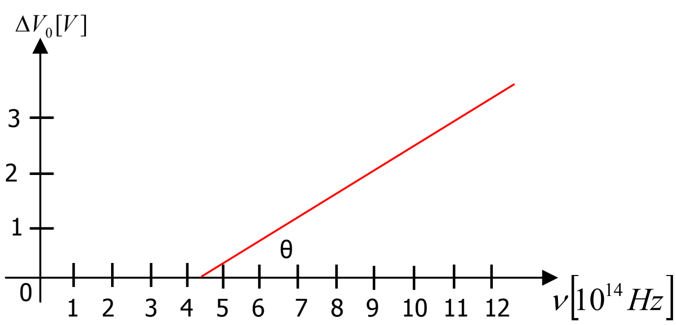

Fig. 2.4: Stop voltage-frequency of the incident radiation.

$tg\theta$ is fixed and is always: $tg\theta = h/e$. The equation of this line, known as Einstein Relation, is, of course:

$E_K = -e\Delta V_0 = h\nu - L_e = \frac{1}{2}m_e V^2$, where L_e is the extraction energy needed for the electron, $h\nu$ is the energy brought from the photon to the electron and $E_K = \frac{1}{2}m_e V^2$ is the kinetic energy with which the electron comes out.

The big news, here, is the relation $E = h\nu$ (Planck/Einstein relation) through which light brings energy: it depends on the frequency through a constant $h = 6{,}625 \cdot 10^{-34} Js$ (Planck's constant).

Par. 2.2: Planck's Black Body Spectrum.

<u>preamble on Boltzmann's Distribution Law:</u>

now we try to understand how changes, in a material, the number of molecules per unit of volume, when the energy changes.

Suppose to have a column of gas at a constant temperature, in a container and under the effect of the gravitational field.

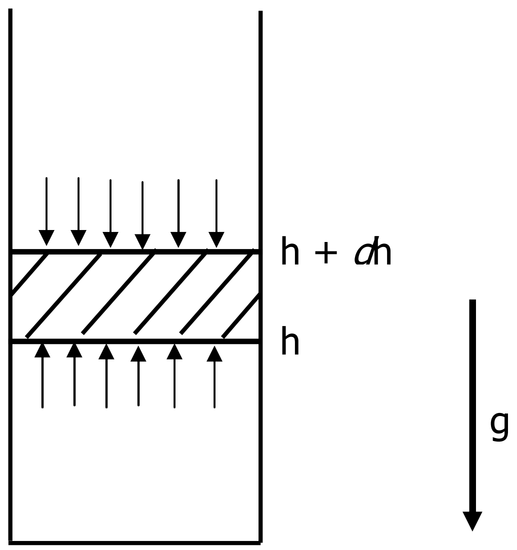

Fig. 2.5: Column of gas.

If this container has a volume V in which we have N gas particles, we define **n** as the number of particles per unit of volume.

With reference to the above figure, we examine a section S of the column of gas at the height h. The pressure P_h at the height h is obviously higher than that at the height h+dh, as at h the mass of gas pushing downwards is higher.

Being pressure P defined as dF/dS =(weight of the disc dh high and section S) / S, we have:

$$P_{h+dh} - P_h = dP = \frac{-m \cdot n \cdot S \cdot dh \cdot g}{S} = -mgndh, \tag{2.1}$$

where m is the mass of every single particle of gas, n is the number of particles per unit of volume, S dh is the volume of the disc, g is the gravitational acceleration end the negative sign tells us that dP is negative (P goes down while we go up).

We also know from thermodynamics that:

$$PV = n_{kmoles}RT = n_{kmoles}N_A \cdot \frac{R}{N_A}T = N \cdot k \cdot T \,, \tag{2.2}$$

where the first equality is the law of ideal gases (R=const), N_A is the number of particles in a kilomole, i.e. the Number of Avogadro, N= n_{kmoles} N_A= is the total number of gas particles (made of n_{kmoles}) and k=R/N_A is the Boltzmann's constant.
For a proof of the equation of state of ideal gases, see any of the books on general Physics, or see the link: http://vixra.org/pdf/1201.0001v1.pdf
From the previous equation, we have:

$$P = \frac{N}{V}kT = nkT \,.$$

By differentiating this equation, we get:

$$dP = dnkT \tag{2.3}$$

By eq. (2.1) and (2.3) , we have:

$$\frac{dn}{n} = -\frac{mg}{kT}dh = -\frac{dE_p}{kT} \,,$$

where dE_p=mgdh is the differential of the potential energy of every particle.
The integration of this differential equation easily yields the following result:

$$n = n_0 e^{-E_p/kT} \,, \tag{2.4}$$

where n_0 is constant.
In case the particles are subject not to the gravitational field, but to any other conservative force, F_i (for instance, the intermolecular forces themselves), which we suppose is oriented along x, in (2.4), instead of the potential energy E_p, we'll have the corresponding potential energy E_i coming from the force F_i, that is:

$$E_i = -\int F_i \cdot dx \,.$$

Finally:

$$n = n_0 e^{-Ei/kT} \tag{2.5}$$

The situation with non conservative forces is here not taken into account, as in this case it wouldn't be even possible to claim the thermal equilibrium.
In our opinion, the Boltzmann's equation (2.5) can be considered as proved and we want to remind you of what it means:
the probability to find molecules in a certain spatial disposition changes exponentially with the opposite of the potential energy of that disposition, divided by kT.

Preamble on the linear harmonic oscillator:
We consider a mass fixed to one end of a spring; the other end is fixed to a wall.
When the mass starts oscillating, as F=ma and, by Hooke, F=-kx, we can write the following differential equation:

$$ma + kx = m\frac{d^2x}{dt^2} + kx = 0$$, whose solution is:

$$x = x_0 \sin(\omega t + \theta) \,, \tag{2.6}$$

where $\omega = \sqrt{k/m}$.

Now, we write the expression for the total energy E (which is the sum of the kinetic energy with the elastic potential one) of such an oscillating mass:

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}kx^2=E=E_k+E_p; \qquad (2.7)$$

This is true because:

$$E_p=-\int F\cdot dx=\int kx\cdot dx=\frac{1}{2}kx^2.$$

Using (2.6) in (2.7) and taking into account the expression for ω , we get:

$$E=\frac{m}{2}\omega^2x_0{}^2\cos^2(\omega t+\theta)+\frac{1}{2}kx_0{}^2\sin^2(\omega t+\theta)= \qquad (2.8)$$

$$=\frac{1}{2}kx_0{}^2[\cos^2(\omega t+\theta)+\sin^2(\omega t+\theta)]=\frac{1}{2}kx_0{}^2$$

As, from the previous expression, kinetic and potential components are the same, we have justified the reason why we assigned two identical values (½)kT for the total energy of the oscillators in the cavity of a black body.

<u>preamble on standing waves:</u>

If a wave S1 propagates in a limited mean, the superposition of it with its reflected one S2 generates a standing wave S:

$S_1=A\sin(kx-\omega t)$, $\qquad$ $S_2=A\sin(kx+\omega t)$.

The difference in sign in the arguments is due to the fact that those waves propagate in opposite directions; moreover, the term $\omega t=2\pi\nu t$ tells us that if we fix a point x, we have an oscillation in time, while the term kx tells us that, if we fix a time t, we see an oscillation by moving along x.

Therefore, a propagating wave oscillates in time and also along the space through which it's propagating indeed.

$$S=S_1+S_2=2A\cdot\sin kx\cdot\cos\omega t=2A\cdot\sin\frac{2\pi}{\lambda}x\cdot\cos 2\pi\nu t; \qquad (2.9)$$

after that we take into account the following trigonometric equality:

$$\sin\alpha+\sin\beta=2\cos\frac{(\alpha-\beta)}{2}\cdot\sin\frac{(\alpha+\beta)}{2}.$$

<u>Planck's Black Body Spectrum:</u>

Let's consider a cavity whose sides are at temperature T, uniform and constant.

Microscopic charges which makes the sides move because of the thermal agitation and, so doing, they radiate electromagnetic waves which fill the cavity; there is an energy transfer from the cavity sides to the electromagnetic field. Simultaneously, electromagnetic waves move into the cavity and hit the sides; so doing, they transfer energy from the field to the cavity sides. An equilibrium is so settled.

The **black body radiation spectrum** is the function $f(\nu)$ so that $f(\nu)d\nu$ is the energy had by the electromagnetic field in the unity of volume of the cavity, and with frequency between ν and $\nu+d\nu$, that is:

$$f(\nu)d\nu=du \qquad [J/m^3]$$

Cavity sides emit and absorb radiation and can be held as made by small oscillating dipole. Moreover, we can assign the radiation in the cavity two degrees of freedom corresponding

to two polarization planes which are perpendicular and independent each other and on which every electromagnetic wave can oscillate; in simpler words, an electromagnetic wave which propagates along z can oscillate transversally on both planes zx and zy.
We know from the kinetic theory of gases that for every particle, and so for every em wave emitted by the particles, and for every degree of freedom we can assign an energy equal to twice $\frac{1}{2}kT$, that is kT, as the total energy is made of a kinetic part and a potential part and their mean values are the same (see (2.8)).
For a proof of the fact that the total energy to be conferred is really kT/degree of freedom see the link: http://vixra.org/pdf/1201.0001v1.pdf
Now, suppose we have, out of simplicity, a cubic cavity whose electromagnetic radiation propagates along the three axis, so generating standing waves; moreover, we consider just one polarization plane per propagation axis (y), and we'll later take into account the real existence of two degrees of freedom.

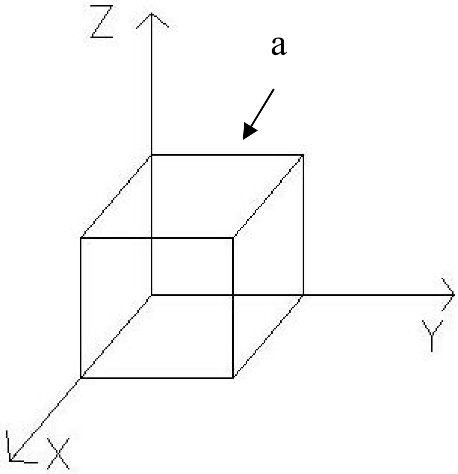

Fig. 2.6.

As the cavity is place of standing waves, and considering the x axis as the propagation one, we will write the following equation for a standing wave (see (2.9)):
$E_y(x,t) = E_{oy}\sin(kx)\cdot\sin(2\pi\nu t)$, k is the wave number = $2\pi/\lambda$ and λ is the wavelength.
We remind that : $c = \lambda\nu$, and: $\omega = 2\pi/T = 2\pi\nu$.
As the standing wave must be zero in x = 0 and in x = a , we have:

$$ka = n\pi \rightarrow n = 2a/\lambda \rightarrow \nu = \frac{c}{\lambda} = \frac{c\cdot n}{2a}.$$

n is positive and not zero, otherwise we don't have any wave.
In general, for a wave propagating along a random direction, we have, component by component:

$$E_y(x,t) = E_{oy}\sin(k_x x)\cdot\sin(2\pi\nu t) \qquad k_x = (2\pi/\lambda)\cdot\cos\alpha$$

$$E_z(y,t) = E_{oz}\sin(k_y y)\cdot\sin(2\pi\nu t) \qquad k_y = (2\pi/\lambda)\cdot\cos\beta$$

$$E_x(z,t) = E_{ox}\sin(k_z z)\cdot\sin(2\pi\nu t) \qquad k_z = (2\pi/\lambda)\cdot\cos\gamma$$

where the three direction cosines are the components of the versor $\hat{k}$ which indicates the direction of propagation of the wave.

Still by analogy with the single dimension case, we have:

$$k_x a = n_x \pi \qquad \rightarrow (2a/\lambda)\cos\alpha = n_x$$
$$k_y a = n_y \pi \qquad \rightarrow (2a/\lambda)\cos\beta = n_y$$
$$k_z a = n_z \pi \qquad \rightarrow (2a/\lambda)\cos\gamma = n_z$$

$$n_x^{\ 2} + n_y^{\ 2} + n_z^{\ 2} = (2a/\lambda)^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 4a^2/\lambda^2,$$
from which:
$$\nu = \frac{c}{\lambda} = \frac{c}{2a}\sqrt{n_x^{\ 2} + n_y^{\ 2} + n_z^{\ 2}} \qquad (2.10)$$
With all values of n, we have all possible ways of vibration. If we put such values n_x , n_y , n_z on three axes and considering the example n_x , n_y , n_z = (1 , 2 , 2) , we see that the number of possible vibrations corresponding to terns n_x , n_y , n_z (n_x , n_y , n_z $\neq 0$, or we have a singularity case) are the vertexes of the following graph, where the n values are different from zero, so they are all the red spots.

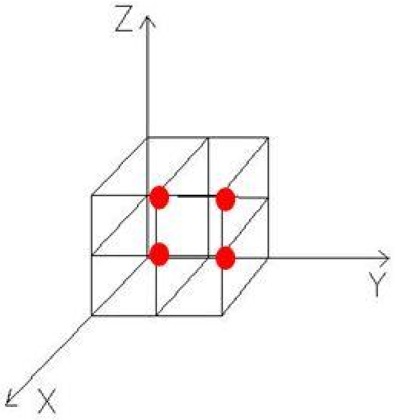

Fig. 2.7.

The fundamental thing we must take into account now (and this has a general validity) is that such *possible ways of vibration (●) correspond, in number, to the small unit side cubes (which are four, too).*

So: ***n . of possible ways of vibration= total volume V located by the tern*** n_x , n_y , n_z .

The radical in the expression (2.10) is just the radius of an octant of sphere located by the three components n_x , n_y , n_z (of course, we consider just the octant where n_x , n_y , n_z are positive, as those must be positive and not zero).

The last remark makes us use the more suitable polar coordinates:

as the volume of an octant of a sphere is equal to $\frac{1}{8}\frac{4}{3}\pi \cdot r^3$, the number N of modes of possible vibrations for a value of r between 0 and r is:

$$N = \frac{1}{8}\frac{4}{3}\pi \cdot r^3 \ .$$

As a consequence, the number N(r) dr of possible modes of vibration for a value of r between r and r + dr can be obtained by differentiating the previous equation:

$$N(r)dr = \frac{\pi}{2} r^2 dr \cdot$$

Now, let's define an N(ν) so that N(r)dr = N(ν)dν=number of possible modes of vibration for frequencies between ν and ν+dν; we see that, according to (2.10), ν= r c / (2 a) , and by differentiating the last equation, we have:

$d\nu = \frac{c}{2a} dr$; and then we get:

$N(\nu)d\nu = \frac{\pi}{2}\left(\frac{2a}{c}\right)^3 \nu^2 d\nu = \frac{4\pi}{c^3} V \nu^2 d\nu$, where $V = a^3$ = volume of the cavity.

Now, in order to pass from the previous equation to f(ν), and remembering that, according to the definition of f(ν) itself we gave before, we have to:

-divide by V to refer to the unity of volume

-multiply by two to take into account the two possible states of polarization of the radiation (as well as we will do when we'll consider the black body)

-multiply by kT, that is, by the mean energy corresponding to each degree of freedom.

Therefore:

$$f(\nu)d\nu = \frac{8\pi}{c^3} kT\nu^2 d\nu \quad , \qquad (2.11)$$

and this equation is known to be the Rayleigh-Jeans equation**.**

Of course:

$$f(\nu) = \frac{8\pi}{c^3} kT\nu^2$$

The graph of this equation is here below:

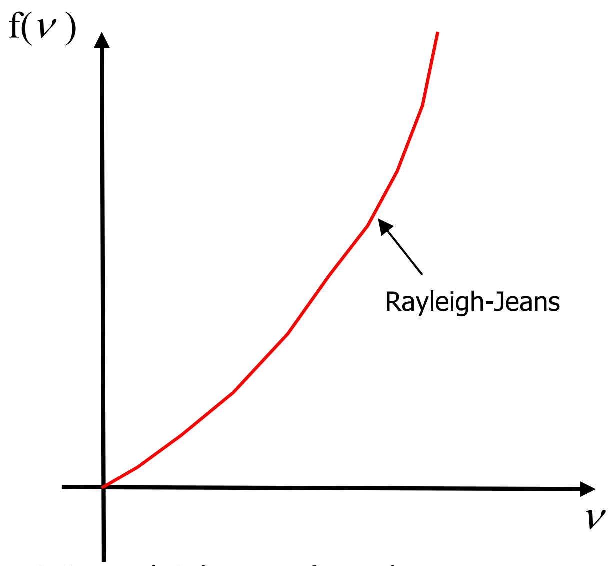

Fig. 2.8: Rayleigh-Jeans' graph.

The experiments, on the contrary, show a different behaviour:

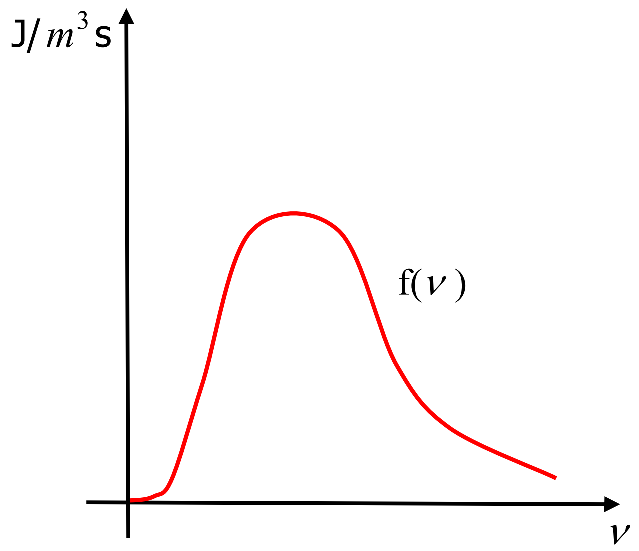

Fig. 2.9: Real emission.

In the real situations, there is a peak, that is a value of frequency around which the emission of the black body concentrates.
Of course, the above curve is for a fixed temperature T and we'll see the more the temperature increases, the higher the frequency values are.
That's why, for instance, a piece of iron at ambient temperature emits an electromagnetic radiation in the range of the infrared waves, or around it, while if you heat it, it will emit visible radiation, at temperatures around some hundreds of centigrade degrees (white heat, red heat).
Similarly, you can find many characteristics of the surface of a star by just studying the frequency spectrum of the light the star irradiates.

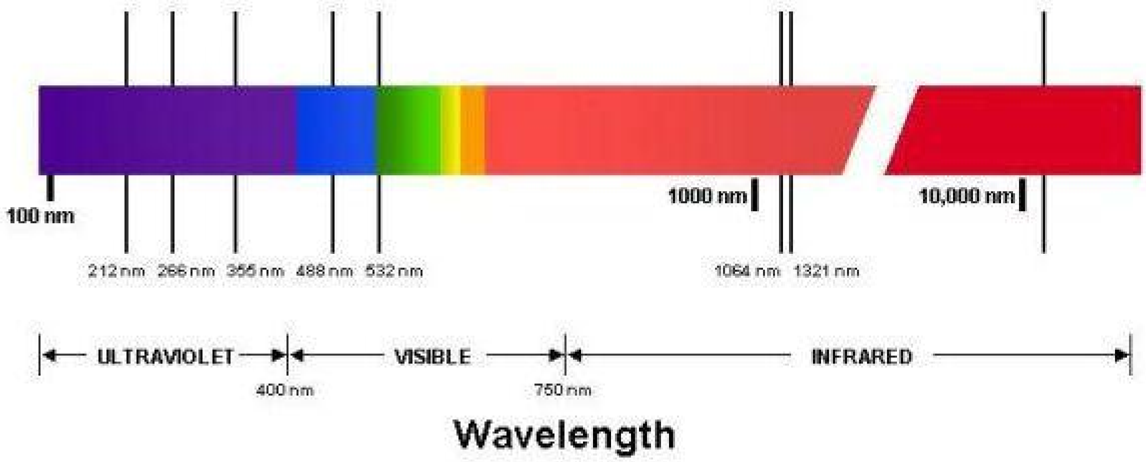

Fig. 2.10: Spectrum of the electromagnetic radiation.

Nothing similar is shown by the Rayleigh-Jeans graph, which leads to an ultraviolet catastrophe. All this was the beginning of the chrisis of classic physics, and there was the need to bring new ideas and quantum hypotheses to make the theoretical deductions match the reality; as an example, we bring the Max Planck's supposition:

first of all, we see that if we want to figure out the mean energy $\overline{E}$ among all energies of the elements of a system, we can carry out a weighed average of all energies, which are distributed according to the already proved Boltzmann's formula (2.5) for n(E); therefore:

$$\overline{E} = \frac{\sum E_i \cdot n(E_i)}{\sum n(E_i)} = \frac{\sum E_i \cdot n_0 e^{-E_i/kT}}{\sum n_0 e^{-E_i/kT}} = \frac{\sum E_i \cdot e^{-E_i/kT}}{\sum e^{-E_i/kT}}; \qquad (2.12)$$

the numerator is the sum of all energies and each of them is weighed according to the number of components which have it, while the denominator is the total number of particles. For the moment, such an average value should be kT, and this is exactly the energy value we conferred to every constituent.

In order to jump from the Rayleigh-Jeans equation to one whose graph is that of the Planck's black body above reported, Planck supposed that for every value of frequency ν, the energy of the system could have just discrete (quantized!) values:

$E = h\nu, 2h\nu, ..., nh\nu$, (n integer). [Planck/Einstein equation]

By such an assumption, (2.12) becomes (summation over n):

$$\overline{E} = \frac{\sum_0^\infty nh\nu \cdot e^{-nh\nu/kT}}{\sum_0^\infty e^{-nh\nu/kT}} .$$

The result is:

In fact, by assuming that $\frac{h\nu}{kT} = z$, we have: $\overline{E} = kT \frac{\sum_0^\infty nz \cdot e^{-nz}}{\sum_0^\infty e^{-nz}}$; by defining:

$f(z) = \sum_0^\infty e^{-nz}$, we have: $-z \cdot df/dz = z\sum_0^\infty n \cdot e^{-nz} = \sum_0^\infty nz \cdot e^{-nz}$, so:

$$\overline{E} = -kTz \frac{df/dz}{f} = -kTz \frac{d}{dz} \ln z = -kTz \frac{d}{dz} \ln \sum_0^\infty e^{-nz} .$$

Now, for Taylor's series, or for the study on geometrical series:

$\sum_0^\infty x^n = \frac{1}{1-x}$, and if we say: $e^{-z} = x$, we have:

$$\overline{E} = -kTz \frac{d}{dz} \ln(1-e^{-z})^{-1} = -kTz \frac{1}{(1-e^{-z})^{-1}} (1-e^{-z})^{-2} e^{-z} = kTz \frac{e^{-z}}{1-e^{-z}} = \frac{kTz}{e^z - 1} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

that is, the assumption , after that we have taken into account the expression for z.

Therefore, Planck's news was to put in Rayleigh-Jeans' equation (2.11), the value of $\overline{E}$, just found, instead of the mean energy per component, that is, kT:

$$f(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} d\nu \qquad (2.13)$$

and this is really the Planck's equation.

By dividing both sides by dν, we get an expression for f(ν) which excellently describes the experimental graph above reported on the black body emission!

Par. 2.3: The Stefan-Boltzmann's Law.

We defined the black body as a cavity. Now, let's make a hole to make some radiation (u [J/m^3]) come out from the cavity, as in the figure below:

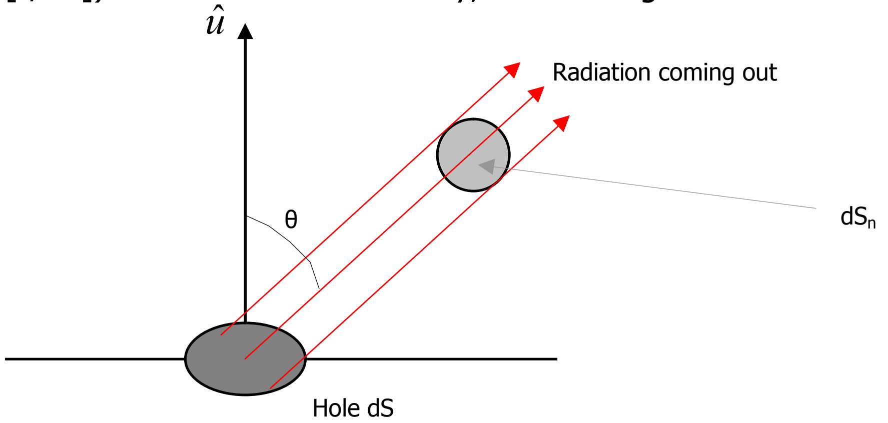

Fig. 2.11: Hole and radiation coming out.

Then, we had, through (2.11) that:

$$f(\nu)d\nu = \frac{8\pi}{c^3}kT\nu^2 d\nu \qquad [J/m^3] \qquad (2.14)$$

If now we introduce the power W [J/s=W] and the solid angle Ω [sr], we easily have, about dS_n:

$$(\frac{d^2W}{d\Omega dv})d\Omega dv = du \cdot c \cdot dS_n \frac{d\Omega}{4\pi} \qquad [W]$$ (power in the interval dv and $d\Omega$.) (2.15)

as such watt on dS_n are due to du [J/m^3] which comes out of the hole by speed c, which is the speed of the radiation, and get [Jm/(m^3s)]= [W/m^2], then again by square meters of dS_n (and we get watt), but all by the fraction of solid angle (dimensionless fraction) $\frac{d\Omega}{4\pi}$ under which dS_n is seen.

We remind, now, that $dS_n = dS\cos\theta$ and $du = f(v)dv$, and (2.15) becomes:

$$(\frac{d^2W}{d\Omega dv})d\Omega dv = f(v)dv \cdot c \cdot dS\cos\theta \frac{d\Omega}{4\pi} \qquad [W]$$

If now we introduce the intensity of radiation, that is I [W/m^2], we have, of course:

$$(\frac{d^2I}{d\Omega dv})d\Omega dv = \frac{1}{dS}(\frac{d^2W}{d\Omega dv})d\Omega dv = \frac{cf(v)}{4\pi}\cos\theta \cdot d\Omega dv \qquad [W/m^2]$$

(the cosine law just seen is the Lambert's Cosine Law).

If now we remind a solid angle can be expressed as a function of polar coordinates angles in the following way: $d\Omega = \sin\theta d\theta d\varphi$, we also have:

$$(\frac{d^2I}{d\Omega dv})d\Omega dv = \frac{1}{dS}(\frac{d^2W}{d\Omega dv})d\Omega dv = \frac{cf(v)}{4\pi}\cos\theta \cdot d\Omega dv = \frac{cf(v)}{4\pi}\cos\theta \cdot \sin\theta \cdot d\theta d\varphi dv \qquad [W/m^2]$$

By integrating this equation over $d\Omega$, that is, over $d\theta d\varphi$ (θ between 0 and π) (φ between 0 and π), and considering that:

$\int_0^{\pi} \cos\theta \sin\theta d\theta = 2\int_0^{\pi/2} \cos\theta \sin\theta d\theta = 2(1/2) = 1$, while the integral over φ is obviously π, we have, in the end:

$$\iint_{\theta-\varphi} (\frac{d^2 I}{d\Omega d\nu}) d\Omega d\nu = (\frac{dI}{d\nu}) d\nu = dI = \frac{cf(\nu)}{4\pi} \cdot 1 \cdot \pi \cdot d\nu = \frac{cf(\nu)}{4} d\nu$$

but as $\frac{dI}{d\nu} = \varepsilon(\nu) \quad [W/(Hz \cdot m^2)] = [J/m^2]$, we have:

$$\varepsilon(\nu)d\nu = (\frac{c}{4}) f(\nu) d\nu \quad [W/m^2] \tag{2.16}$$

Now, through (2.14) and the following one: $\nu = c/\lambda$, we have:

$$\varepsilon(\nu)d\nu = (\frac{c}{4}) f(\nu) d\nu = (\frac{c}{4}) \frac{8\pi}{c^3} kT\nu^2 \cdot d\nu = \frac{2\pi}{c^2} \nu^2 kT d\nu$$

Now, by differentiating $\nu = c/\lambda$, we easily have: $d\nu = c \cdot d\lambda / \lambda^2$ and defining $f(\lambda)$ and $\varepsilon(\lambda)$ as follows (of course):

$f(\lambda)d\lambda = f(\nu)d\nu$

$\varepsilon(\lambda)d\lambda = \varepsilon(\nu)d\nu$

we'll have:

$$f(\lambda)d\lambda = f(\nu)\frac{d\nu}{d\lambda} d\lambda = \frac{8\pi}{c^3} kT\nu^2 \cdot \frac{d\nu}{d\lambda} d\lambda = \frac{8\pi}{c^3} kT\nu^2 \cdot \frac{c}{\lambda^2} d\lambda = \frac{8\pi}{\lambda^4} kT \cdot d\lambda \tag{2.17}$$

$$\varepsilon(\lambda)d\lambda = \varepsilon(\nu)\frac{d\nu}{d\lambda} d\lambda = \frac{2\pi}{c^2} \nu^2 kT \frac{d\nu}{d\lambda} d\lambda = \frac{2\pi}{c^2} \nu^2 kT \frac{c}{\lambda^2} d\lambda = \frac{2\pi c}{\lambda^4} kT \cdot d\lambda \tag{2.18}$$

If now, as well as we did with (2.11) to get (2.13), in (2.17) and (2.18) we put, in place of kT, the expression: $\frac{h\nu}{e^{h\nu/kT} - 1}$, we'll have the following versions of the Planck's Equation:

$$f(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} d\nu \qquad [J/m^3] \tag{2.19}$$

$$f(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/kT\lambda} - 1} d\lambda \qquad [J/m^3] \tag{2.20}$$

$$\varepsilon(\nu)d\nu = \frac{2\pi\nu^2}{c^2} \frac{h\nu}{e^{h\nu/kT} - 1} d\nu \qquad [W/m^2] \tag{2.21}$$

$$\varepsilon(\lambda)d\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/kT\lambda} - 1} d\lambda \qquad [W/m^2] \tag{2.22}$$

Then, by integrating (2.21), we have:

$$\varepsilon = \frac{2\pi h}{c^2} \int_0^{\infty} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu = \frac{2\pi h}{c^2} \int_0^{\infty} \frac{e^{-h\nu/kT}\nu^3}{1 - e^{-h\nu/kT}} d\nu = \frac{2\pi h}{c^2} \int_0^{\infty} [\nu^3 e^{-h\nu/kT} \sum_0^{\infty} (e^{-h\nu/kT})^n] d\nu = \frac{2\pi h}{c^2} \sum_1^{\infty} \int_0^{\infty} \nu^3 e^{-n(h\nu/kT)} d\nu$$

If now we put: $b = \frac{h}{kT}$ and $a = \frac{2\pi h}{c^2}$, we have again:

$$\varepsilon = a\sum_1^{\infty} (-\frac{d^3}{d(bn)^3}) \int_0^{\infty} e^{-bn\nu} d\nu = a\sum_1^{\infty} (-\frac{d^3}{d(bn)^3})(\frac{1}{bn}) = \frac{6a}{b^4} \sum_1^{\infty} \frac{1}{n^4} = \frac{6a}{b^4} \frac{\pi^4}{90} = \frac{2\pi^5 k^4}{15c^2h^3} T^4 = \boxed{\sigma T^4 = \varepsilon}$$

$[W/m^2]$ (Stefan-Boltzmann's Law)

where $\sigma = \frac{2\pi^5 k^4}{15c^2h^3} = 5{,}670 \cdot 10^{-8} \frac{W}{m^2K^4}$ (Stefan-Boltzmann's constant)

In order to prove that $\sum_1^\infty \frac{1}{n^4}$ yields a number equal to $\frac{\pi^4}{90}$ you can just sum the first terms of that series.

Par. 2.4: The Wien's Law.

From (2.22) we have: $\varepsilon(\lambda) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/kT\lambda}-1}$; with reference to Fig. 9, here shown:

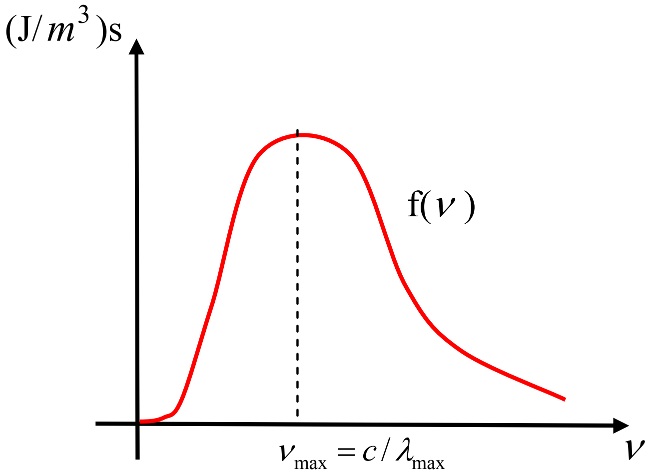

Fig. 2.12: The maximum frequency.

if we want to understand through what λ_{max} the emission takes place, by mathematical analysis we put $\frac{d\varepsilon(\lambda)}{d\lambda} = 0$, that is, we put to zero the first derivative, so:

$5\lambda^4(e^{hc/kT\lambda}-1) + \lambda^5 e^{hc/kT\lambda}(-\frac{hc}{kT\lambda^2}) = 0$, so: $5\lambda e^{hc/kT\lambda} - 5\lambda - \frac{hc}{kT}e^{hc/kT\lambda} = 0$, so, again:

$\frac{(e^{hc/kT\lambda}-1)}{e^{hc/kT\lambda}} = 1 - e^{-hc/kT\lambda} = \frac{hc}{5kT\lambda}$; this trascendental equation, if numerically solved, but also graphically solved, if you like, yields: $\frac{hc}{kT\lambda} = 4{,}965$, from which:

$$\lambda_{max} = \frac{C}{T} = \frac{hc}{k\cdot 4{,}965}\frac{1}{T} = \boxed{\frac{0{,}2897\cdot 10^{-2}}{T} = \lambda_{max}} \quad [m] \quad \text{(Wien's Law)} \qquad (2.23)$$

and $C = 0{,}2897\cdot 10^{-2}\ [K\cdot m]$ is the Wien's Constant.

Par. 2.5: The Compton Effect.

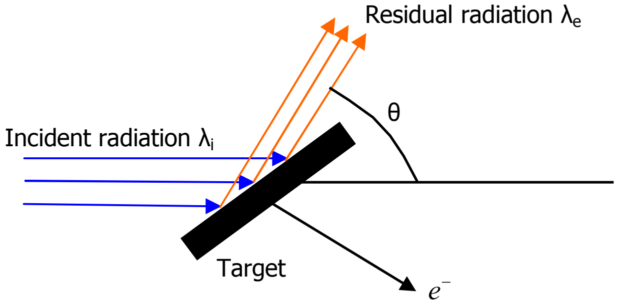

Fig. 2.13: The Compton Effect.

We are here in a situation similar to that of the Photoelectronic Effect; but here, on the contrary, the incident radiation on the target has a very small wavelength λ_i , equal to some tenths of an Å. Therefore, we are talking about very energetic photons.
The electrons will have a certain angle θ′, but we'll see also a residual radiation at λ_e. Being this a very energetic collision, as much as the kinetic energy of the electron can be compared with its rest one $m_e c^2$, it will be held as well as a collision of a photon against a free electron, as if it weren't linked to its nucleus. And we'll have to use the relativistic formulas anyway.
Such an effect, of course, cannot be understood on a classic physics basis.
Now, we show that: $\lambda_e = \lambda_i + \lambda_c(1-\cos\theta)$ (2.24)

$\lambda_c = \dfrac{h}{m_e c} = 0{,}025$ Å is the Compton's wavelength.

Now, we show the vectorial composition of the linear momenta involved:

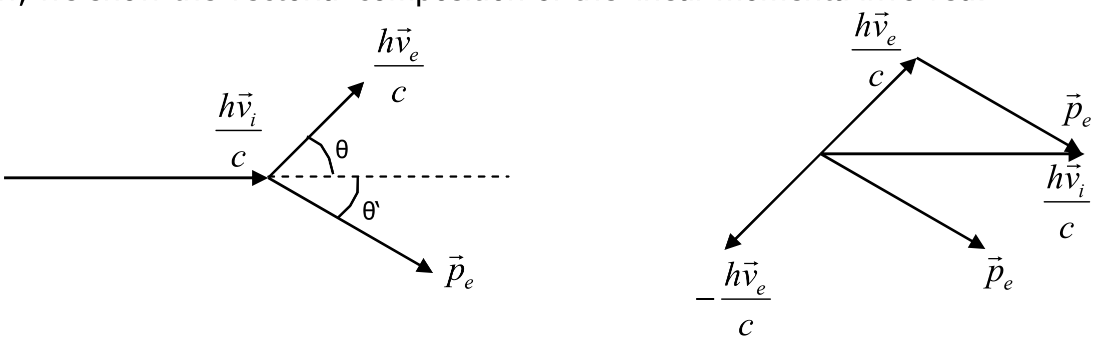

Fig. 2.14: Vectorial composition in the Compton Effect.

We have $\vec{p}_e = \dfrac{h\vec{v}_i}{c} - \dfrac{h\vec{v}_e}{c}$, so, by scalarly multiplying side to side with itself: $\vec{p}_e \cdot \vec{p}_e = \dfrac{h\vec{v}_i}{c}\cdot\dfrac{h\vec{v}_i}{c} + \dfrac{h\vec{v}_e}{c}\cdot\dfrac{h\vec{v}_e}{c} - 2\dfrac{h\vec{v}_i}{c}\cdot\dfrac{h\vec{v}_e}{c}$, that is:

$$p_e^2 = (\frac{hv_i}{c})^2 + (\frac{hv_e}{c})^2 - 2\frac{hv_i}{c}\frac{hv_e}{c}\cos\theta \quad (2.25)$$

Moreover, because of the energy conservation:

$$E_0 + hv_i = E + hv_e \quad (2.26)$$

Now, about the rest quantities, we have: $E_0 = m_e c^2$, (2.27)
$\vec{p}_0 = 0$, while, about the dynamic ones:

$$E = \frac{m_e c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_e c^2 \text{ and} \quad (2.28)$$

$$\vec{p}_e = \frac{m_e \vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_e \vec{v} \quad (2.29)$$

and moreover, from relativity and from the two previous equations, we have:

$$c^2 \vec{p}_e^{\,2} - E^2 = -m_e c^4 \quad (2.30)$$

Now, multiply (2.25) by c^2 and in (2.26) isolate E and then square, so getting:

$$c^2 p_e^2 = (hv_i)^2 + (hv_e)^2 - 2h^2 v_i v_e \cos\theta$$

$E^2 = (E_0 + h\nu_i - h\nu_e)^2 = m_e c^4 + (h\nu_i)^2 + (h\nu_e)^2 + 2m_e c^2 (h\nu_i - h\nu_e) - 2h\nu_i h\nu_e$
and by subtracting side to side those two equations and taking into account (2.30),

$$h\nu_e - h\nu_i = \frac{h^2 \nu_i \nu_e}{m_e c^2}(1-\cos\theta) \qquad (2.31)$$

and now, by multiplying by $\frac{c}{\nu_i \nu_e}(=\frac{\lambda_e}{\nu_i}=\frac{\lambda_i}{\nu_e})$, we get: $\lambda_i - \lambda_e = \frac{h}{m_e c}(1-\cos\theta)$ and (2.24) has been proved.

Now, we calculate θ′ by projecting the already introduced equation $\frac{h\vec{\nu}_i}{c} = \frac{h\vec{\nu}_e}{c} + \vec{p}_e$ on axes; we have:

$0 = \frac{h\nu_e}{c}\sin\theta + p_e \sin\theta'$ and $\frac{h\nu_i}{c} = \frac{h\nu_e}{c}\cos\theta + p_e\cos\theta'$, that is:

$-\frac{h\nu_e}{c}\sin\theta = p_e\sin\theta'$ and $\frac{h\nu_i}{c} - \frac{h\nu_e}{c}\cos\theta = p_e\cos\theta'$

and by dividing side to side, we have: $tg\theta' = \frac{\nu_e \sin\theta}{\nu_i - \nu_e\cos\theta} = \frac{\sin\theta}{\frac{\nu_i}{\nu_e} - \cos\theta}$, but for the (2.31):

$\frac{\nu_i}{\nu_e} = 1 + \frac{h\nu_i}{m_e c^2}(1-\cos\theta)$, so, finally: $tg\theta' = \frac{\sin\theta}{(1+\frac{h\nu_i}{m_e c^2})(1-\cos\theta)} = \frac{\cot(\theta/2)}{(1+\frac{h\nu_i}{m_e c^2})}$.

Chapter 3: A more formal treatise on Quantum Mechanics.

Par. 3.1: The Schrödinger's Equation (formal deduction).

We know the Planck/Einstein's Equation:

$$E = h\nu \qquad (3.1)$$

And we also know the relation between pulsation (angular velocity) ω and frequency ν:

$$\omega = 2\pi\nu \qquad (3.2)$$

Then, for the energy of a particle:

$$E = m_0 c^2 = \vec{p}\cdot\vec{c} \qquad (3.3)$$

and then the linear momentum:

$$\vec{p} = m_0 \vec{c} \qquad (3.4)$$

and, moreover, the general relations $c = \lambda\nu$ (velocity is wavelength by frequency) $|\vec{k}| = \frac{2\pi}{\lambda}$ (modulus of the wave vector $\vec{k} = \frac{2\pi}{\lambda}\hat{k}$) and $\hbar = \frac{h}{2\pi}$ (Dirac's constant – barred $\hbar$).

Now, from (3.1) and (3.3), we have: $p = h\frac{\nu}{c} = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$ (3.5)

Moreover: $E = h\nu = \frac{h}{2\pi}2\pi\nu = \hbar\omega$. (3.6)

And for a particle, $E = \frac{1}{2}mv^2 = \frac{1}{2m}m^2v^2 = \frac{p^2}{2m}$ (3.7)

and $E = \hbar\omega = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$. (3.8)

Now, as in order to locate a particle I have to interfere with it, by illuminating it, or perturbing it somehow, and as, simply speaking, the smaller a particle is, the more that perturbation disturbs it, diverts it, slows sit down, accelerates it etc, one is led not to imagine anymore it as a single point, but rather through a wave.
With De Broglie, we can associate a wavelength to a particle, through (3.5):
$\lambda = \frac{h}{p} = \frac{h}{mV}$, where, now, V is the velocity of the particle and p is the modulus of $\vec{p} = m_0\vec{V}$.
For what has been just said, we are also led to introduce a wave function $\Psi = \Psi(\vec{r},t) = \Psi(\vec{x},t)$ which describes the particle when moving along $\vec{r}(x,y,z)$ (or $\vec{x}(x,y,z)$).

<u>wave function</u>:
for all what previously said, the particle isn't anymore a dimensionless point, but rather something like a cloud which is the space in which the probability to find the particle is higher; if we put $\rho(\vec{x},t)d^3x$ the probability to find the particle in the volume between $\vec{x}$ and $\vec{x}+d^3x$ (d^3x as we are thinking in three dimensions), it must be proportional, through a proportionality constant, to the square modulus $|\Psi(\vec{x},t)|^2$ of the wave function $\Psi = \Psi(\vec{x},t)$. We are talking here about a square modulus, as, in general, we can express a wave through trigonometric functions, and so also in a complex form, that is, with complex numbers and we have quantifiable quantities in the real field, as long as we take their moduli:
$|\Psi(\vec{x},t)|^2 d^3x = |N|^2 \rho(\vec{x},t)d^3x$ ($|\Psi(\vec{x},t)|^2 = \Psi(\vec{x},t)\Psi^*(\vec{x},t)$), where $\Psi^*(\vec{x},t)$ is the complex conjugated of $\Psi(\vec{x},t)$, (i swapped with –i).
Ψ is typical of every single electron. Now, by the definition of probability, the integration over all the space must yield the maximum probability:
$\int \rho(\vec{x},t)d^3x = 1$, so: $\int |\Psi(\vec{x},t)|^2 d^3x = |N|^2$
Let's normalize the function Ψ so that $\int |\Psi(\vec{x},t)|^2 d^3x = 1$, and we have:

$$\Psi_N(\vec{x},t) = \frac{1}{N}\Psi(\vec{x},t)$$

Let's write down a list of some of the properties Ψ must have:

-it must be continuous, as the probability to find the particle, for instance, in x_0, must be the same, whatever you tend to x_0, whether from left or from right.

-it must be limited everywhere, as well as the probability to find the particle in a certain place is.

-for a particle which is localized in a region Ω, we must have $\Psi = 0$ for $x \notin \Omega$.

-it must be a monodrome function (just one value)

-wave functions which differs just by the normalization describe the same physical system (and $\Psi = 0 \rightarrow$ Vacuum)

-if a system can stay in a state Ψ_1 and also in a state Ψ_2 , then it can stay also in a generic state $\Psi = \alpha\Psi_1 + \beta\Psi_2$.

wave function of a free particle:
we know from wave physics that, of course, a wave propagating through time and through x, must have, as an argument, a function like:
$\frac{2\pi}{\lambda}\hat{k}\cdot\vec{x} - \frac{2\pi}{\lambda}vt = \vec{k}\cdot\vec{x} - \omega t$, as if we fix a point in time (as: t=0) we have a variability with x and fixing x we have a variability in time, that is a real wave.
Now, according to (3.5) and (3.6) we have: $\vec{k}\cdot\vec{x} - \omega t = \frac{\vec{p}}{\hbar}\vec{x} - \frac{E}{\hbar}t$ and so the wave function must be like:

$$f(\vec{k}\cdot\vec{x} - \omega t) = f(\frac{\vec{p}}{\hbar}\vec{x} - \frac{E}{\hbar}t) \quad (3.9)$$

We notice that deriving (3.9) over t means to factor ω, while deriving it over x means to factor k.
Now, as according to (3.8): $\omega = \frac{\hbar k^2}{2m}$, we understand, for all what has been just said, that we have to take a t-first order wave equation which is also an x-second order:

$$\frac{\partial\Psi}{\partial t} = \gamma\frac{\partial^2\Psi}{\partial x^2}. \quad (3.10)$$

Now, Fourier should suggest to propose base functions as candidates to be solutions of (3.10), the following four:

$$A\sin(\vec{k}\cdot\vec{x} - \omega t) \quad (3.11)$$

$$B\cos(\vec{k}\cdot\vec{x} - \omega t) \quad (3.12)$$

$$Ce^{i(\vec{k}\cdot\vec{x} - \omega t)} \quad (3.13)$$

$$De^{-i(\vec{k}\cdot\vec{x} - \omega t)} \quad (3.14)$$

So, we notice that (3.11) and (3.12), in their monodimensional form, (x in place of $\vec{x}$ etc), cannot satisfy (3.10), while (3.13) and (3.14) can, provided that we consider:
$-i\omega = -\gamma k^2$, from which: $\gamma = i\frac{\omega}{k^2} = i\hbar\frac{\hbar\omega}{\hbar^2 k^2} = i\hbar\frac{E}{p^2} = \frac{i\hbar}{2m}$ and we notice that γ is here independent from dynamic quantities as p, therefore it works for us.
If, on the contrary, if we chose the d'Alembert wave equation $\frac{\partial^2\Psi}{\partial^2 t} = \gamma\frac{\partial^2\Psi}{\partial x^2}$ (not ok), all four candidates should have satisfied it, but for γ we would have had:
$\gamma = \frac{\omega^2}{k^2} = (\frac{\hbar\omega}{\hbar k})^2 = \frac{E^2}{p^2} = \frac{p^2}{4m^2}$, not ok, as such a γ should be a dynamic parameter, as it has p inside, so such an equation would have changed its characteristics with p.
So, we put (3.13) in our good candidate (3.10), so getting:
$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}$, and, after multiplying both sides by $i\hbar$:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \quad (3.15)$$

(Schrödinger's Equation for a free particle and on a monodimensional motion)

If now we put the expression for $\Psi(x,t)$ ((3.13) monodimensional) in (3.15), we get:

$\hbar\omega\Psi = \frac{\hbar^2 k^2}{2m}\Psi$, that is:

$$E\Psi = \frac{p^2}{2m}\Psi \quad ; \qquad (3.16)$$

in fact, we already had: $E = \frac{p^2}{2m}$.

Now, we rewrite, one over another, (3.15) and (3.16):

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

$$E\Psi = \frac{p^2}{2m}\Psi$$

By a comparison side to side, we see that it is possible to make the following associations of operators:

$E \to i\hbar\frac{\partial}{\partial t}$ and $p^2 \to -\hbar^2\frac{\partial^2}{\partial x^2}$ >>> $p \to -i\hbar\frac{\partial}{\partial x}$

In three dimensions, (3.15) becomes:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi \quad , \qquad (3.17)$$

which is the three-dimension Schrödinger's equation for a free particle, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian, then $\Psi(\vec{x},t) = Ce^{i(\vec{k}\cdot\vec{x}-\omega t)}$, $|\Psi(\vec{x},t)| = C$, $E \to i\hbar\frac{\partial}{\partial t}$, $p^2 \to -\hbar^2\Delta$, $p \to -i\hbar\nabla$, $\vec{p} = \hbar\vec{k}$, $\omega = \frac{\hbar k^2}{2m}$, con $k = |\vec{k}|$.

We notice that the velocity of the wave is $v_f = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m}$, that is, a phase velocity, while the particle velocity is $v_g = \frac{p}{m} = \frac{d\omega}{dk} = \frac{d}{dk}\frac{\hbar k^2}{2m} = 2v_f$, and so it is a group velocity.

Now, as in (3.17) the quantity $-\frac{\hbar^2}{2m}\Delta$ has got the dimension of an energy E, a kinetic one, in this case, and this quantity corresponded to:

$$-\frac{\hbar^2}{2m}\Delta \to \frac{p^2}{2m} = \frac{1}{2m}m^2v^2 = E_k , \qquad (3.18)$$

if the particle is also in a potential V, we'll have, in place of the mere kinetic energy, the total energy $H=T+V=E_k+V$ (H is the Hamiltonian) and (3.17) will become: ($\Psi(\vec{x},t) = Ce^{i(\vec{k}\cdot\vec{x}-\omega t)}$, wave function and $\Psi^*(\vec{x},t) = Ce^{-i(\vec{k}\cdot\vec{x}-\omega t)}$ is its complex conjugated)

$$i\hbar\frac{\partial\Psi}{\partial t} = (-\frac{\hbar^2}{2m}\Delta + V)\Psi$$ **Complete Schrödinger's Equation!** (3.19)

As an alternative, according to (3.18) we can write:

$$E_k = \frac{p^2}{2m} = H - V = \frac{p^2}{2m} \qquad (3.20)$$

and also:

$$-\frac{\hbar^2}{2m}\Delta\Psi = (H - V)\Psi \qquad (3.21)$$

that is : $\Delta\Psi + \frac{2m}{\hbar^2}(H - V)\Psi = 0$ **An alternative for the complete Schrödinger's Equation!** (3.22)

Regarding phase and group velocities, for a photon, which is monocromatic and follows the d'Alembert equation, those two velocities are the same ($v_f = v_g = c$), and all this shows us once again that Schrödinger's Equation is not the same as the d'Alembert wave equation and for it we have: $v_f \neq v_g$.

The Schrödinger's Equation sounds like a tied wave, standing like. As chance would have it. **Wanna see the Schrödinger's Equation, in the formulation of the (3.22), is a standing wave equation???**

Let's try and see:

first of all, we notice that (3.22) really looks like the equation of standing waves:

$$\frac{\partial^2\Psi}{\partial x^2} + k^2\Psi = 0 \text{; (standing waves equation)} \qquad (3.23)$$

Out of simplicity, we consider (3.22) in a monodimensional form:

$\frac{\partial^2\Psi}{\partial x^2} + \frac{2m}{\hbar^2}(H - V)\Psi = 0$; well, it's exactly the same.

(3.23) is the standing wave equation, indeed; as a matter of fact, if a generic Ψ_1 propagates in a limited mean, the superposition of it with its reflection Ψ_2 makes a standing wave $\Psi = \Psi_1 + \Psi_2$: $\Psi_1 = A\sin(kx - \omega t)$, $\Psi_2 = A\sin(kx + \omega t)$.

The difference in sign in the arguments shows that those two waves propagate in opposite directions; moreover, the term $\omega t = 2\pi\nu t$ tells us that, if you fix a point x, you have an oscillation in time, while the term kx tells us that if you fix a time t, you'll see an oscillation when you move along x.

Ψ, therefore, oscillates in time and along the direction of propagation.

$$\Psi = \Psi_1 + \Psi_2 = 2A\sin kx \cdot \cos\omega t = 2A\sin\frac{2\pi}{\lambda}x \cdot \cos 2\pi\nu t \text{;} \qquad (3.24)$$

after that we have used the following trigonometric identity:

$\sin\alpha + \sin\beta = 2\cos\frac{(\alpha - \beta)}{2} \cdot \sin\frac{(\alpha + \beta)}{2}$.

Now, if you fix t in (3.24), you'll have: $\Psi = const \cdot \sin kx$, from which:

$\frac{\partial^2\Psi}{\partial x^2} = -const \cdot k^2 \sin kx = -k^2\Psi$, from which, again: $\frac{\partial^2\Psi}{\partial x^2} + k^2\Psi = 0$, so the (3.23), that is, the standing wave equation!

Therefore, as a further intuitive proof of the Schrödinger's Equation, we give the following: let Ψ be the wave function; it must withstand the following wave equation: $\frac{\partial^2\Psi}{\partial x^2} + k^2\Psi = 0$;

then we know from the previous pages that $p = \hbar k$, from which: $k^2 = \frac{p^2}{\hbar^2}$ and so:

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{p^2}{\hbar^2}\Psi=0\,. \qquad (3.25)$$

Then, we know through (3.20) that: $H-V=\frac{p^2}{2m}$, and so: $\frac{2m}{\hbar^2}(H-V)=\frac{p^2}{\hbar^2}$ and (3.25) yields: $\frac{\partial^2\Psi}{\partial x^2}+\frac{p^2}{\hbar^2}\Psi=\frac{\partial^2\Psi}{\partial x^2}+\frac{2m}{\hbar^2}(H-V)\Psi=0$ so really the (3.22) monodimensional!

Par. 3.2: The Heisenberg's Indetermination Relations (formal deduction).

preamble on the mean value of an operator:
we know that by (Ψ,Ψ) we mean the following: $\int\Psi^*(\vec{x},t)\Psi(\vec{x},t)d^3x$, which is 1 for normalized Ψ.
Before, we talked about probability P as a function of the space (x or $\vec{x}$) and proportional ro the square modulus of the wave function:
$P\propto|\Psi(\vec{x},t)|^2=\Psi(\vec{x},t)\Psi^*(\vec{x},t)$, where $\Psi^*(\vec{x},t)$ is the complex conjugated of $\Psi(\vec{x},t)$ (i swapped with –i). If then you want to calculate the mean value (over the space) for an operator F, we can use the weighed mean value calculation, where the weight evaluated for every point where you want to calculate the mean value, is $\Psi(\vec{x},t)\Psi^*(\vec{x},t)$:

$$\langle F\rangle=(\Psi,F\Psi)=\int\Psi^*(\vec{x},t)F\Psi(\vec{x},t)d^3x \qquad (3.26)$$

preamble on fundamental commutators:
we define the commutator of the operator A with the operator B: $[A,B]=AB-BA$. Now, in case A and B are just numbers, their commutator will be zero, but if they are operators, then things can be different.
For fundamental commutators, we have:

$[x_i,x_j]=x_ix_j-x_jx_i=0$ (x=position)

$[p_i,p_j]=(-i\hbar\frac{\partial}{\partial x_i})(-i\hbar\frac{\partial}{\partial x_j})-(-i\hbar\frac{\partial}{\partial x_j})(-i\hbar\frac{\partial}{\partial x_i})=0$, (we saw that $p\to-i\hbar\frac{\partial}{\partial x}$).

$[x_i,p_j]=i\hbar\delta_{ij}$;

in fact, if you apply the commutator to an auxiliary and generic operator φ:

$$[x_i,p_j]\varphi=x_i(-i\hbar\frac{\partial\varphi}{\partial x_j})-(-i\hbar\frac{\partial}{\partial x_j})(x_i\varphi)=-i\hbar x_i\frac{\partial\varphi}{\partial x_j}+i\hbar\frac{\partial x_i}{\partial x_j}\varphi+i\hbar x_i\frac{\partial\varphi}{\partial x_j}=i\hbar\delta_{ij}\varphi$$

where δ_{ij} is the Kronecker's Delta, and is 0 if $i\neq j$ and 1 if $i=j$. In fact, as x_i and x_j are ortogonal and linearly independent (as x, y and z are), we really have $\frac{\partial x_i}{\partial x_j}=\delta_{ij}$.

About the commutator $[t,E]$: (as $E\to i\hbar\frac{\partial}{\partial t}$)

$[t,E]\varphi=it\hbar\frac{\partial\varphi}{\partial t}-i\hbar\frac{\partial}{\partial t}(t\varphi)=it\hbar\frac{\partial\varphi}{\partial t}-i\hbar\frac{\partial t}{\partial t}\varphi-it\hbar\frac{\partial\varphi}{\partial t}=-i\hbar\frac{\partial t}{\partial t}\varphi=-i\hbar\varphi$ and so:

$[t,E]=-i\hbar$

preamble on the eigenvalue equation and on deviations:
as x_i is a certain position on a certain axis (for instance, x_1=x, x_2=y, x_3=z), then also Ψ_i is a certain state i, considered as a component i of a wave functio Ψ in a maybe infinite-dimension space i=infinite).

If states "i" exist, where an operator F (which can be simply a real number f) has a well defined value, then we have: $\langle F\rangle_i = f_i$.

F should be an "observable", likely. Then, we know the definition of mean square deviation ΔF for F and we want it becomes zero:

$\Delta F = \sqrt{\langle F^2\rangle_i - \langle F\rangle_i^2} = 0$. We also define the "simple deviation" Δ_F:

$\Delta_F = F - \langle F\rangle_i$. Then, we have:

$\langle \Delta_F^2\rangle_i = \langle (F - \langle F\rangle_i)^2\rangle_i = \langle F^2\rangle_i + \langle F\rangle_i^2 - 2\langle F\rangle_i\langle F\rangle_i = \langle F^2\rangle_i - \langle F\rangle_i^2 = (\Delta F)^2$. Now, the request according to which: $\Delta F = 0$, becomes as follows: $\langle \Delta_F^2\rangle_i = 0 = (\Psi_i, \Delta_F^2\Psi_i) = 0$. And as F is an observable, then hermitian (F*=F), also Δ_F will be hermitian, and so we can write:

$\langle \Delta_F^2\rangle_i = (\Psi_i, \Delta_F^2\Psi_i) = (\Delta_F\Psi_i, \Delta_F\Psi_i) = \int |\Delta_F\Psi_i|^2 d\xi = 0$, from which: $\Delta_F\Psi_i = 0$, that is:

$F\Psi_i = f_i\Psi_i$, which is the eigenvalue equation for F.

preamble on the Schwarz's Inequality:

if we consider the scalar product between two vectors as the projection of one over the other, we have: $\vec{u}\cdot\vec{w} = |\vec{u}||\vec{w}|\cos\theta \le |\vec{u}||\vec{w}| = \sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{w}\cdot\vec{w}} = \sqrt{\vec{u}^2}\sqrt{\vec{w}^2}$, as $\cos\theta \le 1$.

$\vec{u}\cdot\vec{w} \le \sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{w}\cdot\vec{w}}$ is a general form for the Schwarz's Inequality.

If now we go back to our quantum operatorial mean values formalism, we have, from analogy: $|(\Psi, FG\Psi)| \le \sqrt{(\Psi, F^2\Psi)}\sqrt{(\Psi, G^2\Psi)}$, that is, also (by squaring both sides, if we like):

$|(\Psi, FG\Psi)|^2 \le (\Psi, F^2\Psi)(\Psi, G^2\Psi) = (\Psi, FF\Psi)(\Psi, GG\Psi)$ and as F and G are hermitian, we'll also have: $|(F\Psi, G\Psi)|^2 \le (F^*\Psi, F\Psi)(G^*\Psi, G\Psi) = (F\Psi, F\Psi)(G\Psi, G\Psi)$, (3.27)

as, from the definition of (3.26), it's very easy to see that an operator between round brackets can be moved from left to right, with respect to the comma, provided that you turn it into its complex conjugated and if it is hermitian, its complex conjugated is equal to itself.

(3.27) is the Schwarz's Inequality we're interested in.

at last, the Heisenberg's Indetermination Relations:

as now we can well manage with all quantum terminology and formalism, as per all what has been said so far, let's try and evaluate the following expression: $(i\langle [F,G]_\Psi\rangle)^2$, where F and G are hermitian:

$(\langle i[F,G]_\Psi\rangle)^2 = |(\Psi, FG\Psi) - (\Psi, GF\Psi)|^2$, but we can also say that:

$|(\Psi, FG\Psi) - (\Psi, GF\Psi)|^2 \le (|(\Psi, FG\Psi)| + |(\Psi, GF\Psi)|)^2$, as the sum of moduli is for sure not less than the simple difference.

As F and G are hermitian, we can say:

$(\Psi, GF\Psi) = (G\Psi, F\Psi) = (FG\Psi, \Psi) = (\Psi, FG\Psi)^*$ and $(\Psi, FG\Psi) = (F\Psi, G\Psi)$ and so, about the previous equations:

$(\langle i[F,G]_\Psi\rangle)^2 \le 4|(F\Psi, G\Psi)|^2$; then, according to Schwarz:

$|(F\Psi, G\Psi)|^2 \le (F\Psi, F\Psi)(G\Psi, G\Psi)$ and so:

$$(\langle i[F,G]_\Psi\rangle)^2 \le 4(\Psi,F^2\Psi)(\Psi,G^2\Psi) = 4\langle F^2\rangle_\Psi \langle G^2\rangle_\Psi \quad (3.28)$$

Before we said: $\Delta_F = F - \langle F\rangle_\Psi$, and, from analogy: $\Delta_G = G - \langle G\rangle_\Psi$, that is:

$$\begin{cases} \Delta_F = F - \langle F\rangle_\Psi \\ \Delta_G = G - \langle G\rangle_\Psi \end{cases} \quad (3.29)$$

and we also got:: $\langle \Delta_F^2\rangle_\Psi = \langle F^2\rangle_\Psi - \langle F\rangle_\Psi^2 = (\Delta F)^2$ and, still from analogy, then also: $\langle \Delta_G^2\rangle_\Psi = \langle G^2\rangle_\Psi - \langle G\rangle_\Psi^2 = (\Delta G)^2$, that is:

$$\begin{cases} \langle \Delta_F^2\rangle_\Psi = \langle F^2\rangle_\Psi - \langle F\rangle_\Psi^2 = (\Delta F)^2 \\ \langle \Delta_G^2\rangle_\Psi = \langle G^2\rangle_\Psi - \langle G\rangle_\Psi^2 = (\Delta G)^2 \end{cases} \quad (3.30)$$

From (3.29) we have: $[\Delta_F,\Delta_G] = [F,G]$, (3.31)

as, in making $[\Delta_F,\Delta_G]$ explicit, products of F and G with the m.v. cancel each other (while FG and GF don't). Now, in (3.28) let's make a replacement: $F \to \Delta_F$ and $G \to \Delta_G$; we have:

$$(\langle i[\Delta_F,\Delta_G]_\Psi\rangle)^2 \le 4\langle {\Delta_F}^2\rangle_\Psi \langle {\Delta_G}^2\rangle_\Psi \quad (3.32)$$

and also taking into account (3.30) and (3.31), (3.32) changes again:

$(\langle i[F,G]_\Psi\rangle)^2 \le 4(\Delta F)^2(\Delta G)^2$, from which:

$$\boxed{\Delta F \cdot \Delta G \ge \frac{1}{2}\left|\langle i[F,G]_\Psi\rangle\right|} \quad (3.33)$$

which is the Heisenberg's Indetermination Relation.

If we now put F=x and G=p and remembering the preambles on fundamental commutators, from (3.33) we have the famous: $\boxed{\Delta x \cdot \Delta p \ge \frac{\hbar}{2}}$. (if I want to know well the position of an electron, then I have to give up some accuracy on the evaluation of its speed $\propto$ p, and vice versa)

On the contrary, if we put F=t and G=E and still remembering preambles on fundamental commutators, still according to (3.33) we'll have the famous (as well): $\boxed{\Delta E \cdot \Delta t \ge \frac{\hbar}{2}}$.

Thank you for your attention.

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Bibliography:

1) *(M. Alonso & E.J. Finn)* FUNDAMENTAL UNIVERSITY PHYSICS III, Addison-Wesley.

2) *(C. Rossetti)* ISTITUZIONI DI FISICA TEORICA (Intr. alla M.Q.), Levrotto & Bella.

3) *(R. Gautreau & W. Savin)* FISICA MODERNA – Schaum.

4) *HOOKE'S LAW AS A BASIS FOR THE UNIVERSE* http://altrogiornale.org/download/266772/

5) *(R. Feynman)* LA FISICA DI FEYNMAN I-II e III – Zanichelli.

6) *(Lionel Lovitch-Sergio Rosati)* FISICA GENERALE, Elettricità, Magnetismo, Elettromagnetismo Relatività Ristretta, Ottica, Meccanica Quantistica , 3^ Edizione; Casa Editrice Ambrosiana-Milano.

7) *(C. Mencuccini e S. Silvestrini)* FISICA I – Meccanica-Termodinamica, Liguori.

8) *(C. Mencuccini e S. Silvestrini)* FISICA II – Elettromagnetismo-Ottica, Liguori.
